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GENERALIZED RAY OPTICAL ANALYSIS OF REFLECTION FROM A SMOOTH CONVEX CONDUCTING SURFACE

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ABSTRACT

In certain applications involving reflection from smooth convex surfaces, the conventional geometrical optics method (GO) is not sufficiently accurate. In this paper a generalization of the conventional GO formulas is presented and closed-form expressions are derived. The derivation of the generalized geometrical optics(GGO) formulas is based on the Fresnel transformation between two planes, one of which contains the spectral point, while the other passes through the observation point. The method developed herein improves the accuracy of GO without paying the high numerical cost of the physical optics integration. It is important, in particular, for computing the reflection characteristics of numerically specified surfaces.

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1. INTRODUCTION

Most of the diffraction-theoretic approaches to high-frequency scattering from a smooth conducting reflector involve integration over the current induced on the reflector by a given incident field [1]. The most straightforward of these approaches is the Physical Optics (PO) approach, wherein the currents on the reflector are approximated by the currents calculated from the theory of Geometrical Optics (GO). If no characteristics of the integrand can be found that will permit simplification of the PO integral, it is often necessary to employ a "brute force" integration which turns out to be lengthy and laborious. However, if the reflector surface is doubly convex, i.e., non-focusing, the PO integral can be asymptotically evaluated at high frequency using the method of stationary phase. The asymptotic physical optics yields closed-form expressions that are identical to those given by the ray method.

However, in certain applications, the ray method result is not sufficiently accurate, and the higher order corrections to GO are necessary. The classical way to obtain higher order GO terms is to expand the electromagnetic field in a high-frequency asymptotic series of the form

$$\vec{E} = \exp\,[-jks(\vec{r})] \sum_{m=0}^{\infty} (jk)^{-m} \; \vec{e}_m(\vec{r}) \tag{1.1}$$

where s is the phase function and $\{\vec{e}_m\}$ are amplitude vectors. Using ray techniques, Lee [2,3] was able to obtain explicit formulas for calculating the first two orders in (1.1) of the field reflected from a conducting surface in terms of the incident field and the surface properties at the

specular point $\vec{A}$. (See Fig. 1.) However, continuation of the amplitude vectors $\vec{e}_m$ from the specular point $\vec{A}$ $(z=z_0)$ to the observation point $\vec{P}$ entails the use of recursion relations of the form:

$$\vec{e}_0(z) = \vec{e}_0(z_0)\, DF(z,z_0) \qquad (1.2a)$$

$$\vec{e}_m(z) = \vec{e}_m(z_0)\, DF(z,z_0) - 1/2 \int_{z_0}^{z} DF(z,z')\, \nabla^2 e_{m-1}(z')\, dz' \qquad m = 1, 2, 3, \ldots \qquad (1.2b)$$

where $DF(z,z_0)$ in the divergence factor (to be explicitly defined later) and $\vec{e}_m(z)$ and $\nabla^2\vec{e}_m(z)$ are given by

$$\vec{e}_m(z) = [\vec{e}_m\,(x,y,z)]_{x=y=0} \qquad (1.3a)$$

$$\nabla^2\vec{e}_m(z) = [\nabla^2\vec{e}_m(x,y,z)]_{x=y=0} \qquad (1.3b)$$

Thus, although the GO fields are expressed only in terms of the axial coordinate z, the transverse coordinates x and y enter implicitly into the picture through $\nabla^2\vec{e}_{m-1}(z)$.

The integrals in the transport equation (1.2) are not generally amenable to evaluation in closed form. This is simply because the z-dependence of $\nabla^2\vec{e}_{m-1}(z)$ is not generally known.

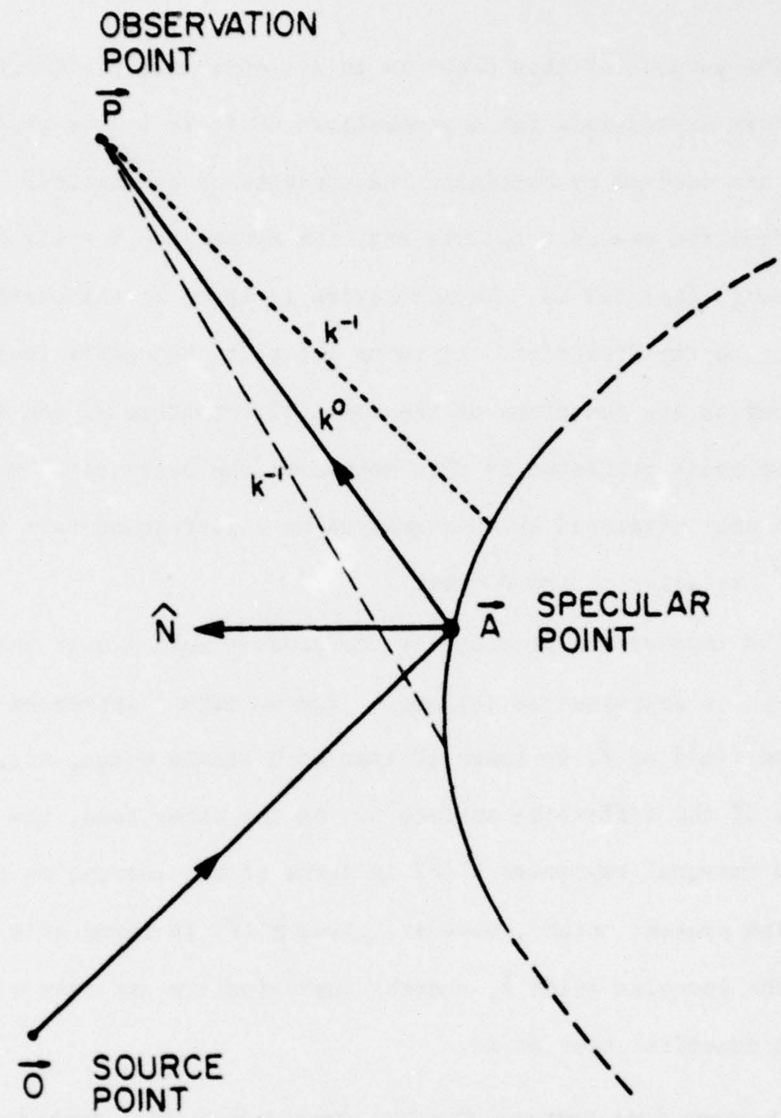

Figure 1. High-frequency scattering of an electromagnetic field from a smooth convex conducting surface. The dotted lines emanating from the neighborhood of the specular point $\vec{A}$ signify higher order (k^{-1}) contributions to the observation point $\vec{P}$.

The purpose of this paper is to overcome this difficulty. Simple closed-form expressions for a generalized GO field at the observation point $\vec{P}$ are derived by combining the concepts of geometrical optics with the diffraction theory. To this end, the expression for the field amplitude is expanded as a Taylor series in terms of the coordinates transverse to the ray direction. It turns out that the coefficients of this Taylor series are functions of the radii of curvature R_1 and R_2 at $\vec{A}$. The field amplitude predicted by this method at the observation point $\vec{P}$ is equal to that predicted by GO augmented by a correction term that is of order k^{-1} relative to the GO term.

The improvement produced by the present method over the classical GO method can be explained as follows. The GO method expresses $\vec{E}^r(\vec{P})$, the reflected field at $\vec{P}$, in terms of that at a single point, viz., the specular point $\vec{A}$, of the reflecting surface Σ. On the other hand, the tedious but more exact PO integral expresses $\vec{E}^r(\vec{P})$ in terms of the current on the whole surface of Σ. The present method, however, gives $\vec{E}^r(\vec{P})$ in terms of a patch of current around the specular point $\vec{A}$, thereby improving the accuracy of GO without paying the high numerical cost of PO.

A comparison between the different approximate methods used in high-frequency electromagnetic scattering is given in Table I.

An important feature of the present method is that it can be associated with the launching method recently developed by Mittra and Rushdi [4] that bypasses the search for the specular point $\vec{A}$, and is suited, in particular, to situations where the reflecting surface Σ is specified only numerically. Consequently, the reflected field computation with generalized GO can be carried out very efficiently.

TABLE I. COMPARISON BETWEEN DIFFERENT APPROXIMATE METHODS USED IN HIGH FREQUENCY ELECTROMAGNETIC REFLECTION

Method	$\vec{E}^r(\vec{P})$ is expressed in terms of field on	Order of k in Field expression
GO	The specular point $\vec{A}$	k^0
GGO	A patch on Σ around $\vec{A}$	k^0 plus k^{-1}
PO	The reflector surface Σ	k^0 plus higher-order terms

lower cost

higher accuracy

2. ON THE AMPLITUDE VARIATION OF THE GO FIELD

This section presents a brief summary of basic GO formulas. It also serves as a clue to the amplitude dependence on transverse coordinates that will be adopted in Section 5.

For the time convention exp [$j\omega t$], a typical scalar component of a ray optical field is given by [5]

$$f = A_{00}(z_0)\ DF(z,z_0)\ \exp[-jks(\vec{r})] \tag{2.1}$$

where $\vec{r} = (x,y,z)$ is the position vector in rectangular pencil coordinates (see Fig. 2). Z coincides with the axial ray which passes through the origin and is measured positively in the direction of wave propagation from $\vec{0} = (0,\ 0,\ z_0)$.

s(r) is the phase function given by

$$s(\vec{r}) = s(\vec{0}) + z + 1/2\vec{\rho}^{\,T}\bar{\bar{Q}}\ \vec{\rho} + 0(|\vec{\rho}|^3) \tag{2.2}$$

$\vec{\rho} = [x \quad y]^T$ gives the transverse coordinates x and y of the pencil. The 2 x 2 matrix $\bar{\bar{Q}}$ is the curvature matrix of the wavefront, given relative to its principal directions by the diagonal form $\bar{\bar{Q}}_D$.

$$\bar{\bar{Q}}_D = \begin{bmatrix} \frac{1}{R_1+z} & 0 \\ 0 & \frac{1}{R_2+z} \end{bmatrix} \tag{2.3}$$

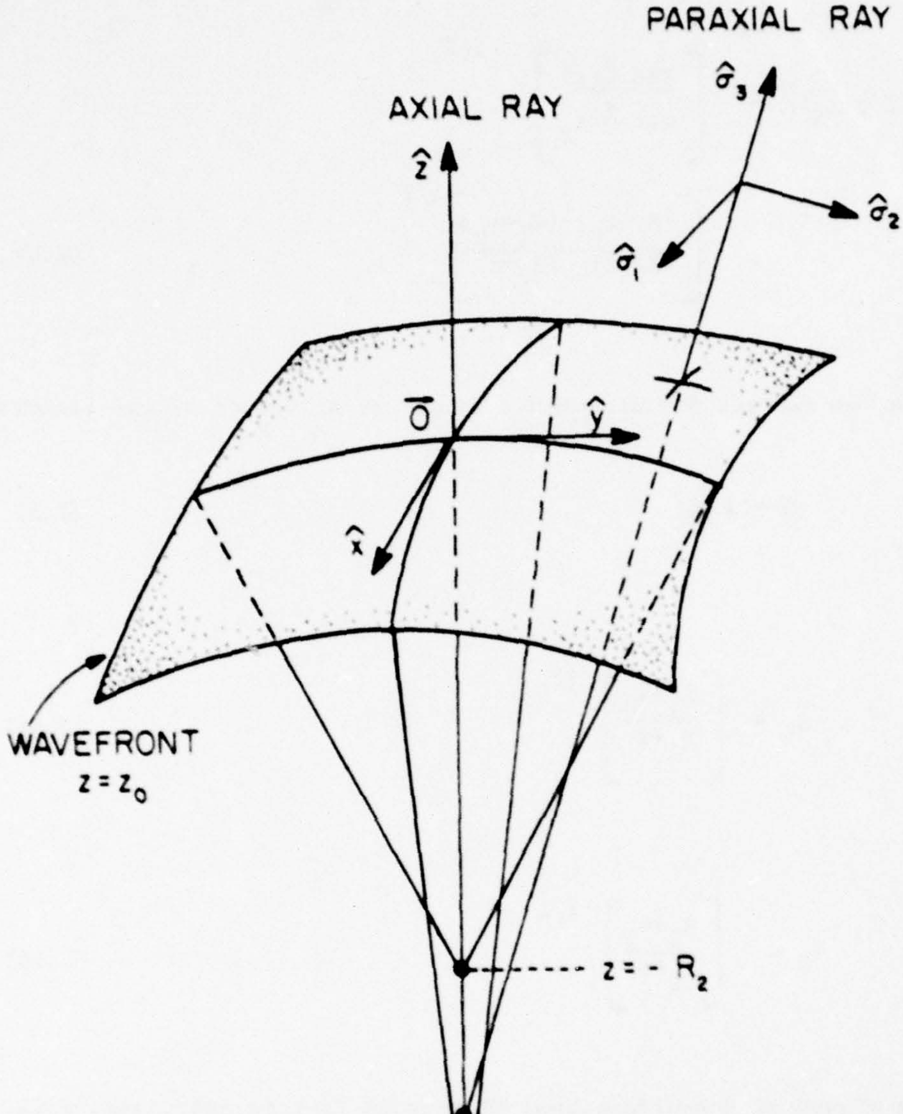

Figure 2. Axial and paraxial rays in a doubly divergent pencil.

where R_1 and R_2 are the principal radii of curvature for the wavefront. $DF(z,z_0)$ is the divergence factor of the pencil given by

$$DF(z,z_0) = \left[\frac{\mathrm{Det}\,\bar{\bar{Q}}(z)}{\mathrm{Det}\,\bar{\bar{Q}}(z_0)}\right]^{1/2}$$

$$= \left[\frac{(R_1+z_0)\ (R_2+z_0)}{(R_1+z)\ (R_2+z)}\right]^{1/2} \tag{2.4}$$

For convenience, we express the divergence factor as a product of two factors

$$DF = a_0 d_0 \tag{2.5}$$

where

$$a_0 = \left[\frac{R_1+z_0}{R_1+z}\right]^{1/2} \tag{2.6a}$$

and

$$d_0 = \left[\frac{R_2+z_0}{R_2+z}\right]^{1/2} \tag{2.6b}$$

a_0 and d_0 can be viewed as one-dimensional divergence factors associated with the principal directions.

Application to electromagnetic waves requires, of course, the inclusion of the polarization and, therefore, an extension of (2.1) to a vector form. The electromagnetic fields $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$ can be written as

$$\vec{E}(\vec{r}) = \vec{e}(\vec{r}) \exp[-jks(\vec{r})] \quad (2.6)$$

$$\vec{H}(\vec{r}) = \vec{h}(\vec{r}) \exp[-jks(\vec{r})] \quad (2.7)$$

where the phase function $s(\vec{r})$ is again given by (2.2) and the amplitude vectors $\vec{e}(\vec{r})$ and $\vec{h}(\vec{r})$ are related via Maxwell's equations by

$$\vec{e} = z_0[(\vec{h} \times \nabla s) - \frac{j}{k}(\nabla \times \vec{h})] \quad (2.8)$$

$$\vec{h} = \frac{1}{z_0}[\nabla s \times \vec{e}) + \frac{j}{k}(\nabla \times \vec{e})] \quad (2.9)$$

where z_0 is the free-space wave impedance. The second terms in (2.8) and (2.9) are neglected under the classical GO approximation.

To resolve the amplitude vectors $\vec{e}(\vec{r})$ and $\vec{h}(\vec{r})$ into longitudinal and transverse components with respect to the paraxial ray through $\vec{r}$, we need to introduce three local orthonormal base vectors at the point $\vec{r}$. With the aid of (2.2) and (2.3), we find the unit vector in the direction of the paraxial ray as

$$\hat{\sigma}_3 = \nabla s$$

$$= \frac{x}{R_1+z}\hat{x} + \frac{y}{R_2+z}\hat{y} + [1 - 1/2\frac{x^2}{(R_1+z)^2} - 1/2\frac{y^2}{(R_2+z)^2}]\hat{z}$$

(2.10)

Since $\hat{\sigma}_3$ is a perturbation of the axial unit vector $\hat{z}$, its component in the $\hat{z}$ direction is given to the order of $|\vec{\rho}|^3$, while its components in the directions transverse to $\hat{z}$ are given to the order of $|\vec{\rho}|^2$ only. This means that $|\hat{\sigma}_3|$ is equal to unity correct to the order of $|\vec{\rho}|^3$.

Now we construct the two orthogonal directions $\hat{\sigma}_1$ and $\hat{\sigma}_2$ transverse to ∇s

$$\hat{\sigma}_1 = [1 - 1/2 \frac{x^2}{(R_1+z)^2}] \hat{x} - 1/2 \frac{xy}{(R_1+z)(R_2+z)} \hat{y}$$

$$- \frac{x}{R_1+z} \hat{z} \tag{2.11}$$

$$\hat{\sigma}_2 = - 1/2 \frac{xy}{(R_1+z)(R_2+z)} \hat{x} + [1 - 1/2 \frac{y^2}{(R_2+z)^2}] \hat{y}$$

$$- \frac{y}{R_2+z} \hat{z} \tag{2.12}$$

The triade $(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ satisfies the orthonormality conditions:

$$(\hat{\sigma}_k \cdot \hat{\sigma}_\ell)^{1/2} = \delta_{k\ell} + O(x_{ij}^3) \qquad k, \ell = 1, 2, 3 \tag{2.13}$$

where $\delta_{k\ell}$ is the kronecker delta and $O(x_{ij}^3)$ denotes terms of the order of $x^\mu y^\nu$ with $\mu + \nu = 3$. For the special case when $\vec{r}$ is on the axial ray $(x=0, y=0)$, the local base vectors $(\hat{\sigma}_1, \hat{\sigma}_2, \nabla s)$ coincide with the pencil base vectors $(\hat{x}, \hat{y}, \hat{z})$. At any point in the pencil, we represent the amplitude vector as

$$\vec{e}(\vec{r}) = e_1\hat{\sigma}_1 + e_2\hat{\sigma}_2 + e_3\hat{\sigma}_3 \tag{2.14}$$

where (e_1, e_2) are the transverse components, and e_3 is the longitudinal component of $\vec{e}$ with respect to the direction of wave propagation. Substituting (2.10), (2.11) and (2.12) into (2.14), we have

$$\vec{e} = \hat{x}\,[(1 - 1/2\,\frac{x^2}{(R_1+z)^2})\,e_1 - 1/2\,\frac{xy}{(R_1+z)\,(R_2+z)}\,e_2$$

$$+ \frac{x}{(R_1+z)}\,e_3\,]$$

$$+ \hat{y}\,[\,-1/2\,\frac{xy}{(R_1+z)\,(R_2+z)}\,e_1 + (1 - 1/2\,\frac{y^2}{(R_2+z)^2})\,e_2$$

$$+ \frac{y}{R_2+z}\,e_3\,]$$

$$+ \hat{z}\,[\,-\frac{x}{(R_1+z)}\,e_1 - \frac{y}{(R_2+z)}\,e_2$$

$$+ (1 - 1/2\,\frac{x^2}{(R_1+z)^2} - 1/2\,\frac{y^2}{(R_2+z)^2}\,)\,e_3]$$

$$= \hat{x}\,e_1' + \hat{y}\,e_2' + \hat{z}\,e_3' \,. \tag{2.15}$$

While the amplitude of the scalar ray optical field (2.1) is independent of the transverse coordinates x and y, Equation (2.15) reveals that the

amplitude vector $\vec{e}$ is a function of the transverse coordinates. If we ignore the transverse dependence of the local components e_1, e_2 and e_3, then the pencil components e_1', e_2' and e_3' are second degree polynomials of $\frac{x}{R_1+z}$ and $\frac{y}{R_2+z}$. Using this result, we will assume a generalized ray optical field of the form:

$$f_0 = \left(\sum_{m,n = 0,0}^{m+n\leq 2} A_{mn}(z)\, x^m y^n \right) \exp\,[-jks(\vec{r})] \qquad (2.16)$$

which reduces to the classical ray optical field when $x = y = 0$.

3. FIELD TRANSFORMATION BETWEEN TWO PLANAR APERTURES.

This section is devoted to a brief presentation of some notations and definitions which are needed for the following two sections. The reader can find a detailed description of the material of this section in [6] and [7]. Let $f_i(\vec{\rho}_i)$ represent a field distribution in the $z = z_i$ plane, whose transverse coordinates x_i, y_i are given by the vector

$\vec{\rho}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$. The Fourier transform pair is

$$F(\vec{\tau}) = F[f(\vec{\rho})] = \iint f(\vec{\rho})\, e^{j\vec{\rho}^T\vec{\tau}}\, d\vec{\rho} \tag{3.1}$$

$$f(\vec{\rho}) = F^{-1}[F(\vec{\tau})] = \frac{1}{(2\pi)^2} \iint F(\vec{\tau})\, e^{-j\vec{\rho}^T\vec{\tau}}\, d\vec{\tau} \tag{3.2}$$

where $\vec{\tau}$ is a two-dimensional vector in the spectral domain, given by

$\vec{\tau} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, and the superscript T on a vector indicates that it is transposed.

The field distribution $f_1(\vec{\rho}_1)$ in the plane $z = z_1$ due to an aperture distribution $f_0(\vec{\rho}_0)$ in the plane $z = z_0$ (see Fig. 3) is given by the Fresnel-Kirchoff integral [6, Eq. 3.5].

$$f_1(\vec{\rho}_1) = \frac{j}{\lambda(z_1-z_0)} \iint f_0(\vec{\rho}_0)\, e^{-jkR}\, d\vec{\rho}_0 \tag{3.3}$$

where λ is the free-space wavelength, $k = \frac{2\pi}{\lambda}$ is the free-space wavenumber, and R is the distance between the points $(\vec{\rho}_0, z_0)$ and $(\vec{\rho}_1, z_1)$, given by

$$R^2 = (z_1 - z_0)^2 + |\vec{\rho}_1-\vec{\rho}_0|^2 . \tag{3.4}$$

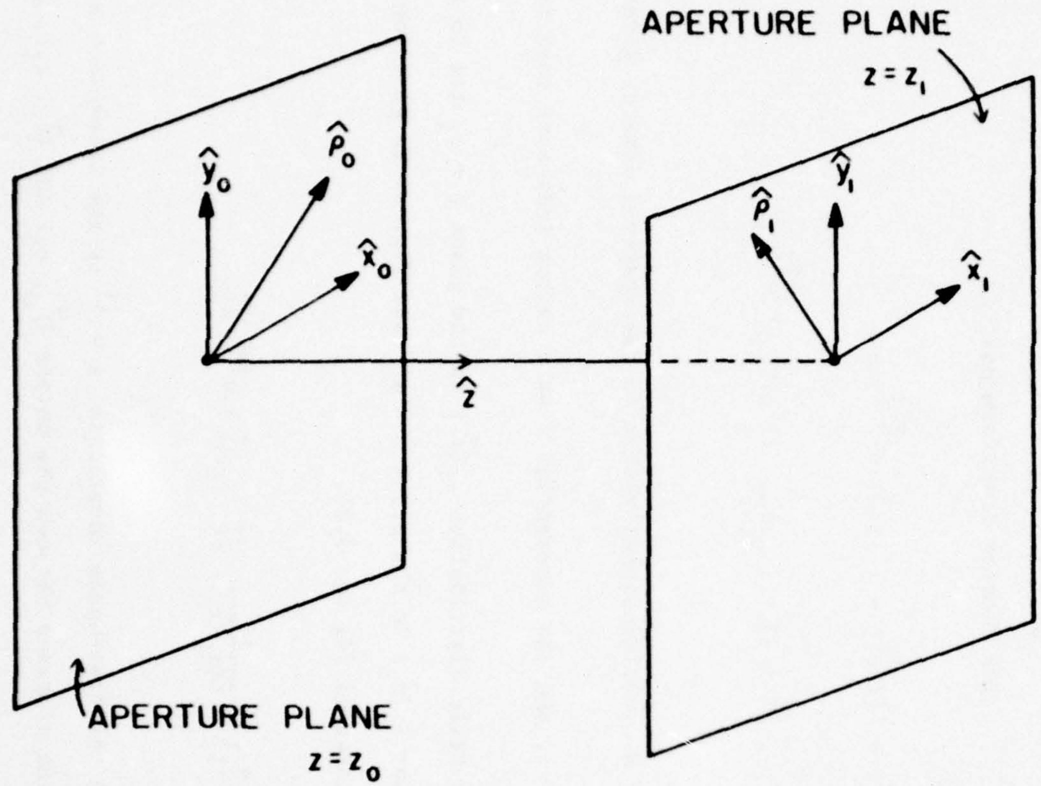

Figure 3. Coordinate system pertinent to the continuation of a field distribution from one plane to another.

If we use the Fresnel approximation

$$(z_1-z_0)^2 \gg |\vec{\rho}_0|^2 \ , \quad (z_1-z_0)^2 \gg |\vec{\rho}_1|^2 \tag{3.5}$$

for all $\vec{\rho}_0$ and $\vec{\rho}_1$ of interest, we can express (3.4) as

$$R \simeq (z_1-z_0) + \frac{|\vec{\rho}_0|^2}{2(z_1-z_0)} + \frac{|\vec{\rho}_1|^2}{2(z_1-z_0)} - \frac{\vec{\rho}_0^{\,T}\vec{\rho}_1}{(z_1-z_0)} \ . \tag{3.6}$$

Substituting (3.6) into (3.3) and making use of (3.1), we finally arrive at

$$f_1(\vec{\rho}_1) = \frac{je^{-jk(z_1-z_0)}}{\lambda(z_1-z_0)} \; e^{-j\frac{k|\vec{\rho}_1|^2}{2(z_1-z_0)}} \; F\left[f_0(\vec{\rho}_0)\, e^{-j\frac{k|\vec{\rho}_0|^2}{2(z_1-z_0)}}\right] \tag{3.7}$$

where the spectral domain vector $\vec{\tau}$ used in the definition of the Fourier transform is given by

$$\vec{\tau} = \frac{k}{(z_1-z_0)} \; \vec{\rho}_1 \ . \tag{3.8}$$

Equation (3.6) can be expressed in an equivalent form of a Fresnel transform [6], but the above form is sufficient for our present purposes. In the next two sections, we will make use of (3.7) to derive the GGO formulas.

4. APPLICATION OF FIELD TRANSFORMATION TO A GO FIELD.

In this section, we will show that if a ray optical field exists at the aperture plane $z = z_0$, then subject to the Fresnel approximation, the field at the aperture plane $z = z_1$ will be that ray field predicted by geometrical optics. This result is important in its own right and serves as a preliminary to the results to be obtained in the following section. From (2.1) and (2.2), a typical scalar component of a ray optical field at an aperture plane $z = z_0$ can be written as

$$f_0(\vec{\rho}_0) = A_{00}(z_0) \exp\left[-jk\left[z_0 + \frac{1}{2}\vec{\rho}_0^{\,T} \bar{\bar{Q}}_0 \vec{\rho}_0\right]\right] \tag{4.1}$$

where $\bar{\bar{Q}}_0$ is the curvature matrix of the wavefront at the plane $z = z_0$. Substituting (4.1) into (3.7) yields

$$f_1(\vec{\rho}_1) = \frac{j e^{-jkz_1} A_{00}(z_0)}{\lambda(z_1 - z_0)} \, e^{-j\frac{k|\vec{\rho}_1|^2}{2(z_1-z_0)}} \, F\left\{ e^{-1/2\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0} \right\} \tag{4.2}$$

where

$$\bar{\bar{H}} = jk\left(\bar{\bar{Q}}_0 + \frac{1}{(z_1 - z_0)} \bar{\bar{I}}\right) \tag{4.3}$$

where $\bar{\bar{I}}$ is the unit matrix, and $\bar{\bar{H}}^{-1}$ is a complex covariance matrix for the zero-mean two-dimensional Gaussian distribution

$\frac{1}{2\pi|\det \bar{\bar{H}}^{-1}|^{1/2}} e^{-1/2\,\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0}$. The Fourier transform of the above distribution is given by [8],

$$F\left\{\frac{1}{2\pi(\det \bar{\bar{H}}^{-1})^{1/2}}\, e^{-1/2\vec{\rho}_0^{\,T}\, \bar{\bar{H}}\, \vec{\rho}_0}\right\} = e^{-1/2\, \vec{\tau}^{\,T}\, \bar{\bar{H}}^{-1}\, \vec{\tau}} \quad . \tag{4.4}$$

Substituting (4.4) into (4.2) and making use of (4.3), we finally arrive at

$$f_1(\vec{\rho}_1) = \frac{A_{00}(z_0)\, e^{-jkz_1}\, e^{-j\frac{k}{2}\vec{\rho}_1^{\,T}\left[\frac{-1}{z_1-z_0}\{(\bar{\bar{Q}}_0(z_1-z_0)+\bar{\bar{I}})^{-1}-\bar{\bar{I}}\}\right]\vec{\rho}_1}}{[\det(\bar{\bar{Q}}_0(z_1-z_0)+\bar{\bar{I}})]^{1/2}} \tag{4.5}$$

where we have made use of

$$(\det \bar{\bar{H}}^{-1})^{-1/2} = (\det \bar{\bar{H}})^{1/2} = j\,\frac{2\pi}{\lambda(z_1-z_0)}\,[\det(\bar{\bar{Q}}_0\,(z_1-z_0) + \bar{\bar{I}})]^{1/2} \tag{4.6}$$

We now proceed to show that expression (4.5) for $f_1(\vec{\rho}_1)$ reduces to that predicted by geometrical optics. In view of (2.3) we obtain:

$$\bar{\bar{Q}}_o(z_1-z_0) + \bar{\bar{I}} = \begin{bmatrix} \frac{R_1+z_1}{R_1+z_0} & 0 \\ 0 & \frac{R_2+z_1}{R_2+z_0} \end{bmatrix} \tag{4.7}$$

Subsequently, we get:

$$[\det(\bar{\bar{Q}}_0(z_1-z_0)+\bar{\bar{I}})]^{1/2} = \left[\frac{(R_1+z_1)\ (R_2+z_1)}{(R_1+z_0)\ (R_2+z_0)}\right]^{1/2}$$

$$= [DF(z_1,z_0)]^{-1} \tag{4.8}$$

and

$$\frac{1}{(z_1-z_0)}\ [(\bar{\bar{Q}}_0\ (z_1-z_0)+\bar{\bar{I}})^{-1}-\bar{\bar{I}}]$$

$$= \begin{bmatrix} \frac{1}{R_1+z_1} & 0 \\ 0 & \frac{1}{R_2+z_1} \end{bmatrix} = \bar{\bar{Q}}_1 \tag{4.9}$$

where $\bar{\bar{Q}}_1$ is the curvature matrix of the wavefront at the plane $z = z_1$. With the aid of (4.8) and (4.9), (4.5) is reduced to

$$f_1(\vec{\rho}_1) = A_{00}(z_0)\ DF(z_1,z_0)\ \exp\ [-jk(z_1+\frac{1}{2}\vec{\rho}_1^{\,T}\ \bar{\bar{Q}}_1\vec{\rho}_1)] \tag{4.10}$$

which is the field distribution predicted by GO. It is interesting to note that (4.9) can be reduced to Equation (3.23) of [5], viz.,

$$\bar{\bar{Q}}_1^{-1} = \bar{\bar{Q}}_0^{-1} + (z_1-z_0)\ \bar{\bar{I}} \tag{4.11}$$

which is the standard equation for phase continuation in geometrical optics.

5. EXTENSION TO GENERALIZED GEOMETRICAL OPTICS FIELD.

In the representation (4.1) of the ray optical field, the amplitude factor $A_{00}(z_0)$ is a function of the axial coordinate z_0 only. In this section, a generalization of that field will be considered, where the amplitude factor is allowed to be a slowly varying function of the transverse coordinates x_0 and y_0, in addition to its z- dependence. This generalized GO field which was suggested by Eq. (2.16) is repeated here for convenience.

$$f_0(\vec{\rho}_0) = \left(\sum_{m,n=0,0}^{m+n\leq 2} A_{mn}(z_0)\, x^m y^n \right) \exp[-jk(z_0 + \tfrac{1}{2} \vec{\rho}_0^{\,T} \bar{\bar{Q}}_0 \vec{\rho}_0)] \; . \tag{5.1}$$

We assume that the field (5.1) exists at the plane $\sigma_0 (z=z_0)$ which is normal to the reflected ray at the specular point $\vec{A}$ in the electromagnetic reflection problem of Section 1. (See Fig. 4). The coefficients $A_{mn}(z_0)$, which are possibly complex, can be expressed in terms of the incident field and the local surface properties at the point $\vec{A}$. Our task here is to use the results of Sections 3 and 4 to find the field distribution $f_1(\vec{\rho}_1)$ at the plane σ_1 $(z=z_1)$ normal to the reflected ray at the observation point $\vec{P}$.

Substituting (5.1) in (3.6) yields:

$$f_1(\vec{\rho}_1) = \frac{je^{-jkz_1}}{\lambda(z_1-z_0)}\, e^{-j\frac{k|\vec{\rho}_1|^2}{2(z_1-z_0)}} \Big\{ A_{00}\, F\{e^{-1/2\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0}\} + A_{10}\, F\{x_0\, e^{-1/2\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0}\}$$

$$+ A_{01}\, F\{y_0\, e^{-1/2\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0}\} + A_{20}\, F\{x_0^2\, e^{-1/2\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0}\}$$

$$+ A_{02}\, F\{y_0^2\, e^{-1/2\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0}\} + A_{11}\, F\{x_0 y_0\, e^{-1/2\vec{\rho}_0^{\,T} \bar{\bar{H}} \vec{\rho}_0}\} \Big\} \tag{5.2}$$

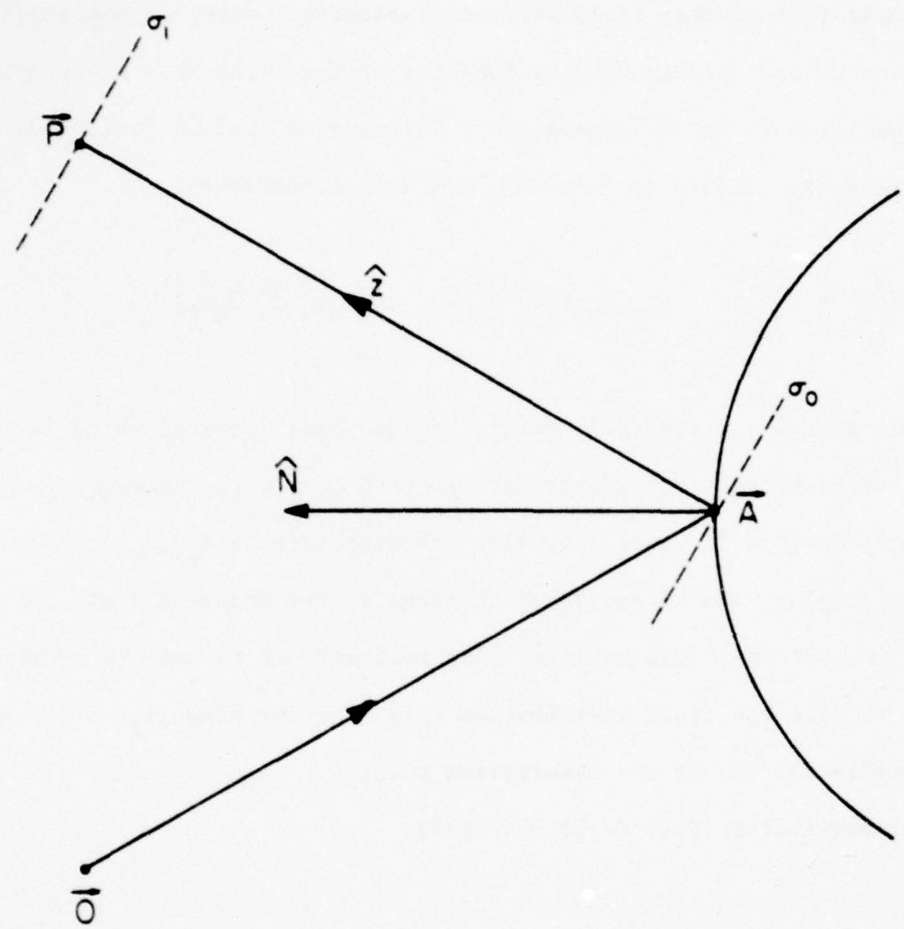

Figure 4. Aperture planes $\sigma_0 (z=z_0)$ and $\sigma_1 (z=z_1)$ normal to the reflected ray at the specular point $\vec{A}$ and the observation point $\vec{P}$, respectively.

where $\bar{\bar{H}}$ is given by Eq. (4.3). Equation (5.2) expresses $f_1(\vec{\rho}_1)$ in terms of six Fourier transforms. The first of these is given by (4.4) as

$$F(\vec{\tau}) = F\{e^{-1/2\,\vec{\rho}_0^{\,T}\,\bar{\bar{H}}\,\vec{\rho}_0}\} = \frac{2\pi}{(\det \bar{\bar{H}})^{1/2}}\, e^{-1/2\,\vec{\tau}^{\,T}\,\bar{\bar{H}}^{-1}\,\vec{\tau}} \tag{5.3}$$

The other five transforms, obtained in Appendix A, are

$$F\left[x_0\, e^{-1/2\,\vec{\rho}^{\,T}\,\bar{\bar{H}}\,\vec{\rho}_0}\right] = \left[a^2 x_1 + 1/2\,(b+c)\,y_1 \quad F(\vec{\tau})\right] \tag{5.4a}$$

$$F\left[y_0\, e^{-1/2\,\vec{\rho}_0^{\,T}\,\bar{\bar{H}}\,\vec{\rho}_0}\right] = \left[1/2\,(b+c)\,x_1 + d^2 y_1\right] F(\vec{\tau}) \tag{5.4b}$$

$$F\left[x_0^2\, e^{-1/2\,\vec{\rho}_0^{\,T}\,\bar{\bar{H}}\,\vec{\rho}_0}\right] = \left[\frac{a^2}{jk}(z_1 - z_0) + (a^2 x_1 + 1/2(b+c)y_1)^2\right] F(\vec{\tau}) \tag{5.4c}$$

$$F\left[y_0^2\, e^{-1/2\,\vec{\rho}_0^{\,T}\,\bar{\bar{H}}\,\vec{\rho}_0}\right] = \left[\frac{d^2}{jk}(z_1 - z_0) + (1/2\,(b+c)\,x_1 + d^2 y_1)^2\right] F(\vec{\tau}) \tag{5.4d}$$

$$F\left[x_0 y_0\, e^{-1/2\,\vec{\rho}_0^{\,T}\,\bar{\bar{H}}\,\vec{\rho}_0}\right] = \left[\frac{(b+c)}{2jk}(z_1 - z_0)\right.$$

$$\left. + (a^2 x_1 + 1/2\,(b+c)\,y_1)\,(1/2\,(b+c)x_1 + d^2 y_1)\right] F(\vec{\tau}) \tag{5.4e}$$

where the nondimensional scalars a^2, b^2, c and d are given by

$$(\bar{\bar{Q}}_0\ (z_1-z_0) + \bar{\bar{I}})^{-1} = \begin{bmatrix} a^2 & b \\ c & d^2 \end{bmatrix} \tag{5.5}$$

Therefore, equation (5.2) for the field distribution at the plane σ_1 can be rewritten as

$$f_1(\vec{\rho}_1) = DF(z_1, z_0)\ \exp\left[-jk\ (z_1 + 1/2\ \vec{\rho}_1^{\,T}\ \bar{\bar{Q}}_1\ \vec{\rho}_1)\right]$$

$$\begin{Bmatrix} A_{00} + \dfrac{(z_1-z_0)}{2jk}\ (2a^2\ A_{20} + 2d^2\ A_{02} + (b+c)\ A_{11}) \\ + (a^2\ A_{10} + 1/2\ (b+c)\ A_{01})\ x_1 \\ + (1/2\ (b+c)\ A_{10} + d^2\ A_{01})\ y_1 \\ + (a^4\ A_{20} + 1/4\ (b+c)^2\ A_{02} + 1/2\ a^2\ (b+c)\ A_{11})\ x_1^2 \\ + (1/4\ (b+c)^2\ A_{20} + d^4\ A_{02} + 1/2\ d^2\ (b+c)\ A_{11})\ y_1^2 \\ + (1/2\ a^2\ (b+c)\ A_{20} + 1/2\ d^2\ (b+c)\ A_{02} + (a^2 d^2 + 1/4\ (b+c)^2)\ A_{11}) \\ x_1 y_1 \end{Bmatrix} . \tag{5.6}$$

Equation (5.6) is the GGO formula we had been seeking and it has a form suitable for computational work. However, additional insight into the physical interpretation of this formula can be gained by writing (5.6) in a special form as follows. If we choose $(\hat{x}_0, \hat{y}_0)$ as the principal directions $(\hat{x}_{0D}, \hat{y}_{0D})$, b and c become zero while a and d reduce to a_0 and d_0 given by Eqs. (2.6a) and (2.6b), respectively. We then have

$$f_1(\vec{\rho}_{1D}) = a_0 d_0 \exp\left[-jk(z_1 + 1/2\, \vec{\rho}_{1D}^{\,T}\, \bar{\bar{Q}}_{1D}\, \vec{\rho}_{1D})\right]$$

$$\left\{ \sum_{m,n=0,0}^{m+n \le 2} A^D_{m,n}(z_0)\, (a_0^2 x_{1D})^m\, (d_0^2\, y_{1D})^n \right.$$

$$\left. -j\, \frac{(z_1 - z_0)}{k}\, (a_0^2\, A^D_{20}(z_0) + d_0^2\, A^D_{02}(z_0)) \right\} . \tag{5.7}$$

We show in Appendix B that (5.6) and (5.7) are equivalent, i.e., the GGO formula is invariant to a rotation of the transverse axes.

Some comments on the GGO formula are in order:

1) Due to the transitivity of the Fresnel transform operator, the expression for $f_1(\vec{\rho}_1)$ in terms of $f_0(\vec{\rho}_0)$ is self-consistent. If we Fresnel-transform either $f_0(\vec{\rho}_0)$ of Eq. (5.1) or $f_1(\vec{\rho}_1)$ of Eq. (5.6) to obtain $f_2(\vec{\rho}_2)$ at $z = z_2$, we arrive at the same result. This is proven in Appendix C.
2) The classical GO amplitude term, viz. A_{00}, is dominant. For a doubly divergent pencil ($R_1 > 0$, $R_2 > 0$), this term decreases

with the pencil advancement as does the divergence factor $a_0 d_0$. The decay of the higher-order amplitude terms is proportional to $a_0^{2m+1}\ d_0^{2m+1}$, hence this decay is higher for a larger m.

3) The amplitude at an axial point is given by (5.8a)

$$A_{00}(z_0) = \frac{j(z_1 - z_0)}{k}\ \left(a_0^2\ A_{20}^D(z_0) + d_0^2\ A_{02}^D(z_0)\right)\ .$$

Substituting for a_0^2 and d_0^2 we can write it explicitly as

$$A_{00}(z_0) = \frac{j(z_1 - z_0)}{k}\ \left[\left(\frac{R_1 + z_0}{R_1 + z_1}\right)\ A_{20}^D(z_0) + \left(\frac{R_2 + z_0}{R_2 + z_1}\right)\ A_{02}^D(z_0)\right] \tag{5.8b}$$

The GGO amplitude result differs from the GO result $A_{00}(z_0)$ by a term of order k^{-1}. For large z_1 and finite R_1 and R_2, the GGO amplitude expression asymptotically approaches

$$A_{00}(z) = jk^{-1}\ \left[(R_1 + z_0)\ A_{20}^D(z_0) + (R_2 + z_0)\ A_{02}^D(z_0)\right] \tag{5.9}$$

4) The correction term in (5.8) can be obtained from the transport equation (1.2b) for m = 1 if we have the following two conditions:

a) $$\left|\frac{\partial^2 \vec{e}_0}{\partial x^2} + \frac{\partial^2 \vec{e}_0}{\partial y^2}\right| >> \left|\frac{\partial^2 \vec{e}_0}{\partial y^2}\right| \tag{5.10}$$

b) The transverse second partial derivatives of $\vec{e}_0$ have the following z-dependence.

$$\frac{\partial^2 \vec{e}_0}{\partial x_D^2} = 2\,\vec{A}^D_{20}(z) = 2\left(\frac{R_1+z_0}{R_1+z}\right)^{5/2}\left(\frac{R_2+z_0}{R_2+z}\right)^{1/2}\vec{A}^D_{20}(z_0) \tag{5.11a}$$

$$\frac{\partial^2 \vec{e}_0}{\partial y_D^2} = 2\,\vec{A}^D_{02}(z) = 2\left(\frac{R_1+z_0}{R_1+z}\right)^{1/2}\left(\frac{R_2+z_0}{R_2+z}\right)^{1/2}\vec{A}^D_{02}(z_0) \tag{5.11b}$$

The type of z-dependence indicated by (5.14) is the same as those indicated by the paraxial-ray analysis (Eq. (2.5)) and by the Fresnel transform analysis (Eq. (5.7)). As pointed out in Section 1, it was the non-availability of the z-dependence of $\nabla^2 e_{m-1}$ that prevented a closed-form evaluation of higher-order GO fields at the observation point.

APPENDIX A

COMPUTATION OF THE FOURIER TRANSFORMS IN EQUATION (5.2)

The first of the Fourier Transforms in Eq. (5.2) is given by

$$F(\vec{\tau}) = F\,[\exp\,(-1/2\;\vec{\rho}_0^{\,T}\;\bar{\bar{H}}\;\vec{\rho}_0)\,]$$

$$= \frac{2\pi}{(\det\bar{\bar{H}})^{1/2}}\;\exp\,(-1/2\;\vec{\tau}^{\,T}\;\bar{\bar{H}}^{-1}\;\vec{\tau})\,. \tag{A.1}$$

Using (4.3) and (5.5), we can express the 2×2 matrix $\bar{\bar{H}}^{-1}$ as

$$\bar{\bar{H}}^{-1} = \frac{z_1 - z_0}{jk}\begin{bmatrix} a^2 & b \\ c & d^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta \\ \gamma & \alpha_2 \end{bmatrix} \tag{A.2}$$

so that (A.1) can be rewritten as

$$F(\vec{\tau}) = \frac{2\pi}{(\det\;\bar{\bar{H}}\;)^{1/2}}\;\exp\,[-1/2\;(\alpha_1\,\xi^2 + (\beta+\gamma)\xi\eta + \alpha_2\,\eta^2)]\;. \tag{A.3}$$

Now, rewriting (3.1) as

$$F(\vec{\tau}) = F[f(\vec{\rho}_0)] = \iint f(\vec{\rho}_0)\;e^{j\,\vec{\rho}_0^{\,T}\vec{\tau}}\;d\vec{\rho}$$

or equivalently

$$F(\xi,\eta) = \iint f(x_0,y_0)\;\exp\,[j(x_o\xi + y_0\eta)]\;dx_0\;dy_0\;, \tag{A.4}$$

we can arrive at expressions for the last five transforms in Eq. (5.2) by differentiating (A.3) or equivalently (A.4) with respect to ξ, η, ξ^2, η^2 and $\xi\eta$, respectively.

$$F\,[x_0 \exp(-1/2\ \vec{\rho}_0^{\,T}\ \bar{\bar{H}}\ \vec{\rho}_0)] = -j\,\frac{\partial F(\xi,\eta)}{\partial \xi}$$

$$= j/2\ [2\,\alpha_1\,\xi + (\beta+\gamma)\,\eta]\ F(\vec{\tau}) \tag{A.5}$$

$$F\,[y_0 \exp(-1/2\ \vec{\rho}_0^{\,T}\ \bar{\bar{H}}\ \vec{\rho}_0)] = -j\,\frac{\partial F(\xi,\eta)}{\partial \eta}$$

$$= j/2\ [(\beta+\gamma)\,\xi + 2\alpha_2\,\eta]\ F(\vec{\tau}) \tag{A.6}$$

$$F\,[x_0^2 \exp(-1/2\ \vec{\rho}_0^{\,T}\ \bar{\bar{H}}\ \vec{\rho}_0)] = -\,\frac{\partial^2 F(\xi,\eta)}{\partial\,\xi^2}$$

$$= [\alpha_1 - 1/4\ (2\alpha_1\,\xi + (\beta+\gamma)\,\eta)^2]\,F(\vec{\tau}) \tag{A.7}$$

$$F\,[y_0^2 \exp(-1/2\ \vec{\rho}_0^{\,T}\ \bar{\bar{H}}\ \vec{\rho}_0)] = -\,\frac{\partial^2 F(\xi,\eta)}{\partial \eta^2}$$

$$= [\alpha_2 - 1/4\ (\ (\beta+\gamma)\,\xi + 2\,\alpha_2\,\eta\,)^2]\ F(\vec{\tau}) \tag{A.8}$$

$$F\,[x_0 y_0 \exp(-1/2\ \vec{\rho}_0^{\,T}\ \bar{\bar{H}}\ \vec{\rho}_0)] = -\,\frac{\partial^2 F(\xi,\eta)}{\partial\xi\partial\eta}$$

$$= [1/2\ (\beta+\gamma) - 1/4\ (2\alpha_1\xi + (\beta+\alpha)\eta)((\beta+\gamma)\ + 2\alpha_2\eta)\ ]\ F(\vec{\tau}) \tag{A.9}$$

The expressions given in (5.4) can be derived by making the following substitutions in (A.5) through (A.9)

$$\alpha_1 = \frac{z_1 - z_0}{jk} a^2 \tag{A.10a}$$

$$\alpha_2 = \frac{z_1 - z_0}{jk} d^2 \tag{A.10b}$$

$$\beta = \frac{z_1 - z_0}{jk} b \tag{A.10c}$$

$$\gamma = \frac{z_1 - z_0}{jk} c \tag{A.10d}$$

$$\xi = \frac{k}{z_1 - z_0} x_1 \tag{A.11a}$$

$$\eta = \frac{k}{z_1 - z_0} y_1 \tag{A.11b}$$

which are the scalar equivalents of (A.2) and (3.8). The results of this appendix can also be derived in a concise manner by taking the first two Fréchet derivatives [10] of both sides of Eq. (5.3) with respect to the vector $\vec{\tau}$.

$$F\,[\vec{\rho}_0\, e^{-1/2\vec{\rho}_0^{\,T}\, \bar{\bar{H}}\, \vec{\rho}_0}] = -j \left(\frac{\partial F(\vec{\tau})}{\partial \vec{\tau}}\right)^T$$

$$= j/2\, [\, \bar{\bar{H}}^{-1} + \bar{\bar{H}}^{-1^T}]\, \vec{\tau}\, F(\vec{\tau}) \tag{A.12}$$

$$F\,[\vec{\rho}_0\vec{\rho}_0^{\,T}\, e^{-1/2\,\vec{\rho}_0^{\,T}\,\bar{\bar{H}}\,\vec{\rho}_0}] = -\frac{\partial F(\vec{\tau})}{\partial \vec{\tau}^2}$$

$$= 1/2\,[\,\bar{\bar{H}}^{-1} + \bar{\bar{H}}^{-1^T}]\;\; F(\vec{\tau})$$

$$- 1/4\,[\,\bar{\bar{H}}^{-1} + \bar{\bar{H}}^{-1^T}]\;\vec{\tau}\vec{\tau}^{\,T}\,[\,\bar{\bar{H}}^{-1} + \bar{\bar{H}}^{-1^T}]\;\; F(\vec{\tau}) \qquad \text{(A.13)}$$

where

$$\vec{\rho}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\vec{\tau} = \frac{k}{(z_1 - z_0)} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$$\vec{\rho}_0\,\vec{\rho}_0^{\,T} = \begin{bmatrix} x_0^2 & x_0 y_0 \\ x_0 y_0 & y_0^2 \end{bmatrix}$$

$$\frac{\partial F(\vec{\tau})}{\partial \vec{\tau}} = \begin{bmatrix} \frac{\partial F}{\partial \xi} & \frac{\partial F}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial^2 F(\vec{\tau})}{\partial \vec{\tau}^2} = \begin{bmatrix} \frac{\partial^2 F}{\partial \xi^2} & \frac{\partial^2 F}{\partial \xi\,\partial \eta} \\ \frac{\partial^2 F}{\partial \xi\,\partial \eta} & \frac{\partial^2 F}{\partial \eta^2} \end{bmatrix}$$

and

$$[\,\underline{\underline{H}}^{-1} + \underline{\underline{H}}^{-1\,T}\,] = \frac{z_1 - z_0}{jk} \begin{bmatrix} 2a^2 & b+c \\ b+c & 2d^2 \end{bmatrix}$$

Equation (A.12) is equivalent to Eqs. (A.5) and (A.6), while Eq. (A.13) is equivalent to Eqs. (A.7), (A.8) and (A.9).

APPENDIX B

INVARIANCE OF THE GGO FORMULA TO THE ROTATION OF THE TRANSVERSE AXES

Consider the case when the transverse directions $(\hat{x},\hat{y})$ do not coincide with the principal directions $(\hat{x}_D, \hat{y}_D)$ of the wavefront but make an angle $-\psi$ with respect to them (Figure 5). The coordinate vectors

$\vec{\rho} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{\rho}_D = \begin{bmatrix} x_D \\ y_D \end{bmatrix}$ are related by

$$\vec{\rho} = \bar{\bar{T}} \quad \vec{\rho}_D \tag{B.1}$$

$$\vec{\rho}_D = \bar{\bar{T}}^T \quad \vec{\rho} \tag{B.2}$$

where the transformation matrix $\bar{\bar{T}}$ is a unitary matrix given by

$$\bar{\bar{T}} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \tag{B.3}$$

where C stands for $\cos\Psi$ and S stands for $\sin\Psi$.

Since the phase quadratic form $1/2\ \vec{\rho}^T\ \bar{\bar{Q}}_0\ \vec{\rho}$ is invariant to rotation of the transverse axes [5], we have

$$1/2\ \vec{\rho}_D^T\ \bar{\bar{Q}}_{0D}\vec{\rho}_D = 1/2\ \vec{\rho}^T\ \bar{\bar{T}}\ \bar{\bar{Q}}_{0D}\ \bar{\bar{T}}^T\ \vec{\rho} = 1/2\ \vec{\rho}^T\ \bar{\bar{Q}}_0\ \vec{\rho}$$

from which we obtain

$$\bar{\bar{Q}}_0 = \bar{\bar{T}}\ \bar{\bar{Q}}_{0D}\ \bar{\bar{T}}^T \tag{B.4}$$

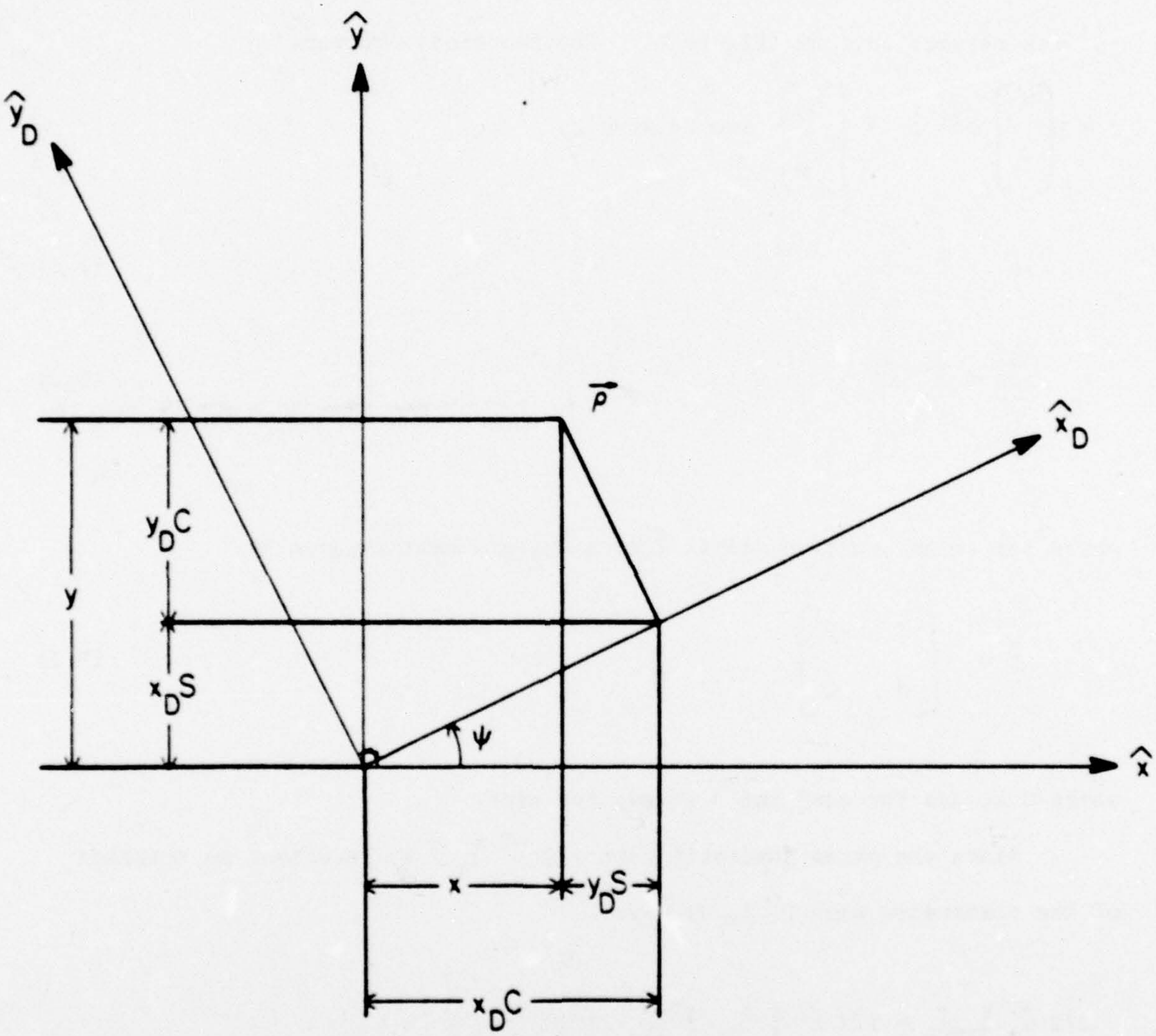

Figure 5. The transverse axes (x, y) make an angle $-\psi$ with the principal axes (x_D, y_D). Here C denotes $\cos\psi$ and S denotes $\sin\psi$.

Using expression (2.3) for $\bar{\bar{Q}}_{0D}$ and expression (B.3) for $\bar{\bar{T}}$, we rewrite $\bar{\bar{Q}}_0$ as

$$\bar{\bar{Q}}_0 = \begin{bmatrix} \frac{C^2}{R_1+z_0} + \frac{S^2}{R_2+z_0} & \left(\frac{1}{R_1+z_0} - \frac{1}{R_2+z_0}\right) SC \\ \left(\frac{1}{R_1+z_0} - \frac{1}{R_2+z_0}\right) SC & \frac{S^2}{R_1+z_0} + \frac{C^2}{R_2+z_0} \end{bmatrix} \tag{B.5}$$

Using (2.4), (2.6) and (B.5) we rewrite (6.21) to (6.23) as

$$a^2 = a_0^2 + S^2 W \tag{B.6}$$

$$d^2 = d_0^2 - S^2 W \tag{B.7}$$

$$b+c = -2\ SC\ W \tag{B.8}$$

where

$$W = (z_1 - z_0) \left[\frac{1}{R_1+z_0} - \frac{1}{R_2+z_0}\right] a_0^2\, d_0^2$$

$$= \left[\left(\frac{z_1-z_0}{R_1+z_0} + 1\right) - \left(\frac{z_1-z_0}{R_2+z_0} + 1\right) \right] a_0^2\, d_0^2$$

$$= \left[\frac{1}{a_0^2} - \frac{1}{d_0^2} \right] a_0^2\, d_0^2 = d_0^2 - a_0^2 \tag{B.9}$$

so that (B.6) to (B.8) can be rewritten as

$$a^2 = C^2 a_o^2 + S^2 d_o^2$$

$$d^2 = S^2 a_o^2 + C^2 d_o^2$$

$$b + c = 2SC\,(a_o^2 - d_o^2)\,.$$

Next we relate the amplitude coefficients in the two coordinate systems. Using (5.1), a scalar component of the amplitude vector can be expressed as

$$h_0 = A_{00} + [A_{10}\quad A_{01}]\ \vec{\rho}_0 + \vec{\rho}_0^{\,T} \begin{bmatrix} A_{20} & 1/2\ A_{11} \\ 1/2\ A_{11} & A_{02} \end{bmatrix} \vec{\rho}_0 \tag{B.10a}$$

$$= A_{00}^D + [A_{10}^D\quad A_{01}^D]\ \vec{\rho}_0^{\,D} + \vec{\rho}_0^{\,D\,T} \begin{bmatrix} A_{20}^D & 1/2\ A_{11}^D \\ 1/2\ A_{11}^D & A_{02}^D \end{bmatrix} \vec{\rho}_0^{\,D} \tag{B.10b}$$

Hence, by making use of (B.1), we arrive at

$$A_{00}^D = A_{00} \tag{B.11a}$$

$$A_{10}^D = C\ A_{10} + S\ A_{01} \tag{B.11b}$$

$$A_{01}^D = -S\ A_{10} + C\ A_{01} \tag{B.11c}$$

$$A_{11}^D = 2\ SC\ (A_{02} - A_{20}) + (C^2 - S^2)\ A_{11} \tag{B.11d}$$

$$A_{20}^D = C^2\ A_{20} + S^2\ A_{02} + CS\ A_{11} \tag{B.11e}$$

$$A_{02}^{D} = S^2 A_{20} + C^2 A_{02} - CS A_{11} \tag{B.11f}$$

Our task now is to show that the scalar amplitude h_1 is the same when expressed in either coordinate system. In view of (5.6) and (5.7), we want to show that

$$h_1^D = DF \left\{ \sum_{m,n=0,0}^{m+n\leq 2} A_{mn}^{D}(z_0) \, (a_0^2 x_{1D})^m \, (d_0^2 \, y_{1D})^n \right.$$

$$\left. - j \, \frac{(z_1 - z_0)}{k} \, (a_0^2 \, A_{20}^{D}(z_0) + d_0^2 \, A_{02}^{D}(z_0) \,) \right\} \tag{B.12}$$

and

$$h_1 = DF \left\{ A_{00} - j \, \frac{(z_1 - z_0)}{k} \, (a^2 A_{20} + d^2 A_{02} + 1/2 \, (b+c) \, A_{11} \,) \right.$$

$$+ [a^2 \, A_{10} + 1/2 \, (b+c) \, A_{01}] \, x_1$$

$$+ [1/2 \, (b+c) \, A_{10} + d^2 A_{01}] \, y_1$$

$$+ [a^4 \, A_{20} + 1/4 \, (b+c)^2 \, A_{02} + 1/2 \, a^2 \, (b+c) \, A_{11}] \, x_1^2$$

$$+ [1/4 \, (b+c)^2 \, A_{20} + d^4 \, A_{02} + 1/2 \, d^2 \, (b+c) \, A_{11}] \, y_1^2$$

$$\left. + [1/2 \, a^2 \, (b+c) \, A_{20} + 1/2 \, d^2 \, (b+c) \, A_{02} + (a^2 d^2 + 1/4 \, (b+c)^2) \, A_{11}] \, x_1 y_1 \right\}$$

(B.13)

are equal.

Substituting (B.2) and (B.11) in (B.12) we get

$$h_1^D = DF \Bigg\{ A_{00} - j \frac{(z_1 - z_0)}{k} \left[(a_0^2 \; C^2 + d_0^2 \; S^2)\; A_{20} \right.$$

$$+ (d_0^2 \; S^2 + a_0^2 \; C^2)\; A_{02} + CS\; (a_0^2 - d_0^2)\; A_{11} \Big]$$

$$+ a_0^2 \; (C\; x_1 + S\; y_1)\; (C\; A_{10} + S\; A_{01})$$

$$+ d_0^2 \; (-S\; x_1 + C\; y_1)\; (-S\; A_{10} + C\; A_{01})$$

$$+ a_0^4 \; (C\; x_1 + S\; y_1)^2 \; (C^2\; A_{20} + S^2\; A_{02} + CS\; A_{11})$$

$$+ d_0^4 \; (-S\; x_1 + C\; y_1)^2 \; (S^2\; A_{20} + C^2\; A_{02} - CS\; A_{11})$$

$$+ a_0^2 \; d_0^2 \; (C\; x_1 + S\; y_1)\; (-S\; x_1 + C\; y_1)(2\; SC\; (A_{02} - A_{20}) + (C^2 - S^2)\; A_{11} \Bigg\} \quad (B.14)$$

which reduces to (B.13) by virtue of (B.6') to (B.8'). This shows that representing the GGO formula relative to the principal axes $(\hat{x}_D, \hat{y}_D)$ is equivalent to representing it relative to any other set of orthogonal axes $(\hat{x}, \hat{y})$.

APPENDIX C

TRANSITIVITY OF THE GGO FORMULA

Due to the transitivity of the Fresnel transform operator [6], the expression for $f_1(\vec{\rho}_1)$ in terms of $f_0(\vec{\rho}_0)$ is self-consistent in the sense that if we Fresnel-transform either $f_0(\vec{\rho}_0)$ of Eq. (5.1) or $f_1(\vec{\rho}_1)$ of Eq. (5.7) to obtain $f_2(\vec{\rho}_2)$ at $z = z_2$, we arrive at the same result. In view of the results in Appendix B, we lose no generality in the following proof if we take the transverse coordinates as the principal ones.

Starting from $f_0(\rho_0)$ and using Eq. (5.7) we can write

$$f_i(\vec{\rho}_{iD}) = a_{0i}\, d_{0i} \exp\left[-jk\left(z_i + 1/2\, \vec{\rho}_{iD}^{\,T}\, \bar{\bar{Q}}_{iD}\, \vec{\rho}_{iD}\right)\right]$$

$$\cdot \left\{ \sum_{m,n=0,0}^{m+n\leq 2} A_{mn}^{D}(z_0)\, (a_{01}^2\, x_{iD})^m\, (d_{01}^2\, y_{iD})^n \right.$$

$$\left. -j\, \frac{(z_i - z_0)}{k} \left(a_{0i}^2\, A_{20}^{D}(z_0) + d_{0i}^2\, A_{02}^{D}(z_0)\right) \right\} \tag{C.1}$$

$$i = 1,2$$

where

$$a_{0i}^2 = \frac{R_1 + z_0}{R_1 + z_i} \tag{C.2a}$$

$$i = 1,2$$

and

$$d_{0i}^2 = \frac{R_2 + z_0}{R_2 + z_i} \tag{C.2b}$$

$$i = 1,2$$

Starting from $f_1(\vec{\rho}_1)$ of Eq. (C.1), we can rewrite $f_2(\vec{\rho}_2)$ as

$$f_2(\vec{\rho}_2) = \left[\frac{(R_1+z_1)}{(R_1+z_2)} \frac{(R_2+z_1)}{(R_2+z_2)} \right]^{1/2} a_{01}\, d_{01} \exp\left[-jk\left(z_2 + 1/2\, \vec{\rho}^{\,T}_{2_D} \bar{\bar{Q}}_D \vec{\rho}_{2_D}\right)\right]$$

$$\cdot \left\{ \sum_{m,n=0,0}^{m+n\le 2} A^D_{mn}(z_0)\, a^{2m}_{01}\, d^{2n}_{01} \left[\frac{R_1+z_1}{R_1+z_2} x_{2_D}\right]^m \left[\frac{R_2+z_1}{R_2+z_2} y_{2_D}\right]^n \right.$$

$$-j \frac{z_1-z_0}{k} \left(a^2_{01} A^D_{20}(z_0) + d^2_{01} A^D_{02}(z_0)\right)$$

$$\left. -j \frac{z_2-z_1}{k} \left[\frac{R_1+z_1}{R_1+z_2} A^D_{20}(z_1) + \frac{R_2+z_1}{R_2+z_2} A^D_{02}(z_1)\right] \right\} \qquad (C.3)$$

which reduces to

$$f_2(\vec{\rho}_2) = a_{02}\, d_{02} \exp\left[-jk\left(z_2 + 1/2\, \vec{\rho}^{\,T}_{2D} \bar{\bar{Q}}_D \vec{\rho}_{2D}\right)\right]$$

$$\cdot \left\{ \sum_{m,n=0,0}^{m+n\le 2} A^D_{mn}(z_0)\, (a^2_{02}\, x)^m\, (d^2_{02}\, y)^n \right.$$

$$- j/k \left[\left((z_1-z_0) + (z_2-z_1) \frac{R_1+z_0}{R_1+z_2} \right) a^2_{01}\, A^D_{20}(z_0) \right.$$

$$\left.\left. + \left((z_1-z_0) + (z_2-z_1) \frac{R_2+z_0}{R_1+z_2} \right) d^2_{01}\, A^D_{02}(z_0) \right] \right\} . \qquad (C.4)$$

To show the agreement of Eq. (C.4) with Eq. (C.2) for $i=2$, we need only to consider the k^{-1} term, which is

$$- \frac{j}{k} \left[\left(\frac{(z_1-z_0)\ (R_1+z_2) + (z_2-z_1)\ (R_1+z_0)}{(R_1+z_2)\ (R_1+z_1)} \right) (R_1+z_0)\ A_{20}^D\ (z_0) \right.$$

$$\left. + \left(\frac{(z_1-z_0)\ (R_2+z_2) + (z_2-z_1)\ (R_1+z_0)}{(R_2+z_2)\ (R_2+z_1)} \right) (R_2+z_0)\ A_{02}^D\ (z_0) \right]$$

$$= - \frac{j(z_2-z_0)}{k} \left[\frac{R_1+z_0}{R_1+z_2}\ A_{20}^D\ (z_0) + \frac{R_2+z_0}{R_2+z_2}\ A_{02}^D\ (z_0) \right]$$

$$= -j\ \frac{(z_2-z_0)}{k}\ \left[a_{02}^2\ A_{20}^D\ (z_0) + d_{02}^2\ A_{02}^D\ (z_0) \right] \tag{C.5}$$

The proof of transitivity of GGO formulas is complete.

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